International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

Some Characterizations of the Underlying Division Ring of a Hilbert Lattice by Automorphisms

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Received July 4, 1997

We give an ortholattice theoretical version, by means of an ortholattice automorphism, of the theorem of M. P. Solèr characterizing Hilbert spaces by orthomodular spaces. Given an orthomodular space $\mathcal{H}$ and an orthoclosed subspace X of $\mathcal{H}$, we study the group of all unitary operators on $\mathcal{H}$ whose restrictions to X and to $X^\perp$ are both identical maps. This enables us to obtain complete characterizations of the underlying division ring of a Hilbert lattice, for each classical case where this division ring is **R**, **C**, or **H** (the skew field of quaternions), by means of one or several ortholattice automorphisms.

1. ORTHOMODULAR SPACES AND HILBERT LATTICES

The results given in this and the next section are well known and most of them can be found in Piron (1976), Varadarajan (1984), Keller (1980), Gross and Künzi (1985) and Solèr (1995).

Let us consider a division ring K equipped with an involutive antiautomorphism denoted by *, and let $(\mathcal{H}, <.,.>)$ be an orthomodular space (also called generalized Hilbert space) over K.

The ortholattice $\mathcal{C}(\mathcal{H})$ of all orthoclosed subspaces X of $\mathcal{H}$, that is, those subspaces X such that $X = X^{\perp\perp}$, is a complete, atomic, irreducible orthomodular lattice satisfying the covering law.

An ortholattice $\mathcal{L}$ isomorphic to such an ortholattice $\mathcal{C}(\mathcal{H})$ is called a *Hilbert lattice;* in the particular case where $\mathcal{H}$ is a classical Hilbert space over **R**, **C**, or **H** (resp. the field of real numbers, the field of complex numbers, and the skew field of quaternions, endowed with their natural conjugation), $\mathcal{L}$ is called a *classical Hilbert lattice.*

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Any orthomodular lattice $\mathcal{L}$ of height at least 4 satisfying the four above properties (complete, atomic, irreducible, satisfying the covering law) is a Hilbert lattice.

It was shown by Keller (1980) and Gross and Künzi (1985) that there exist many nonclassical Hilbert lattices.

Solèr (1995) proved the following outstanding result:

An infinite-dimensional orthomodular space over K is a classical Hilbert space if and only if it contains a γ-orthogonal system, where γ is a nonzero element of K, that is a sequence $(e_n)_{n \varepsilon \mathbf{N}}$ of pairwise orthogonal vectors such that, for any $n \ \varepsilon\ \mathbf{N}$, $<e_n, e_n> = \gamma$.

In that paper, Solèr gave an ortholattice theoretical version of her theorem by means of an "angle-bisecting system."

Our aim is to give another ortholattice theoretical version of Solèr's theorem by means of an ortholattice automorphism, and moreover, in a similar way, to give complete characterizations of the underlying division ring for each of the three classical cases where $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$.

2. SEMIUNITARY AND UNITARY MAPS

In what follows we will suppose that the inversion ring K, with involution, is fixed. We denote by $\mathcal{H}$, $\mathcal{H}'$ orthomodular spaces over K.

Definition 1. A semiunitary map σ: $\mathcal{H} \mapsto \mathcal{H}'$ is a bijective map such that:

(a) For any $x, y \in \mathcal{H}$, $\sigma(x + y) = \sigma(x) + \sigma(y)$.

(b) There exists an automorphism σ' of K such that, for any $\lambda \ \varepsilon\ K$ and any $x \in \mathcal{H}$,

$$\sigma(\lambda x) = \sigma'(\lambda)\sigma(x)$$

(c) There exists $\lambda_\sigma \ \varepsilon\ K$ such that, for any $x, y \ \varepsilon\ \mathcal{H}$,

$$\langle \sigma(x), \sigma(y) \rangle = \sigma'(\langle x, y \rangle)\lambda_\sigma$$

Some Properties

- σ' and λ_σ are unique and are determined by the restriction of σ to any nonzero subspace of $\mathcal{H}$.
- The mapping f defined, for $X \in \mathcal{C}(\mathcal{H})$, by

$$f(X) = \{\sigma(x) | x \in X\}$$

is an ortholattice isomorphism from $\mathcal{C}(\mathcal{H})$ to $\mathcal{C}(\mathcal{H}')$, which is said to be induced by σ.

- By Wigner's theorem, if $\mathcal{H}$ and $\mathcal{H}'$ are of dimension at least 3, every ortholattice isomorphism f from $\mathcal{C}(\mathcal{H})$ to $\mathcal{C}(\mathcal{H}')$ is induced by a semiunitary map.

• If $\mathcal{H}$ and $\mathcal{H}'$ are of dimension at least 2, two semiunitary maps σ_1, σ_2 from $\mathcal{H}$ onto $\mathcal{H}'$ induce the same isomorphism f from $\mathcal{C}(\mathcal{H})$ to $\mathcal{C}(\mathcal{H}')$ if and only if there exists a nonzero element γ of K such that $\sigma_1 = \gamma\sigma_2$.

In particular, if $\mathcal{H}$ is of dimension at least 2, a semiunitary map σ from $\mathcal{H}$ to $\mathcal{H}$ induces the identity on $\mathcal{C}(\mathcal{H})$ if and only if σ is of the form $\gamma id_{\mathcal{H}}$, where γ is a nonzero element of K, and $id_{\mathcal{H}}$ is the identical map on $\mathcal{H}$.

• Any $X \in \mathcal{C}(\mathcal{H})$ equipped with the restriction of the scalar product $<.,.>$ is an orthomodular space over K, and $\mathcal{C}(X)$ is the interval $[0, X]$ of $\mathcal{C}(\mathcal{H})$ endowed with its natural structure of ortholattice inherited from $\mathcal{C}(\mathcal{H})$. Moreover, if $\sigma: \mathcal{H} \mapsto \mathcal{H}'$ is a semiunitary map inducing $f: \mathcal{C}(\mathcal{H}) \mapsto C(\mathcal{H}')$, then the restriction $\sigma|_X$ of σ to X is a semiunitary map from X to $Y = f(X)$ which induces $f|_{[0, X]}: [0, X] \mapsto [0, Y]$.

Definition 2. A semiunitary map $\sigma: \mathcal{H} \mapsto \mathcal{H}'$ is said to be unitary if

$$\forall x, y \in \mathcal{H}, \quad \langle \sigma(x), \sigma(y)\rangle = \langle x, y\rangle$$

or equivalently if $\sigma' = id_K$ and $\lambda_\sigma = 1$.

• It follows from the first property above that, if the restriction of a semiunitary map $\sigma: \mathcal{H} \mapsto \mathcal{H}'$ to a nonzero subspace $X \in \mathcal{C}(\mathcal{H})$ is unitary, then σ is also unitary.

3. A NEW VERSION OF SOLÈR'S THEOREM

Let $(\mathcal{H}, <.,.>)$ be an orthomodular space over K.

Lemma 1. Let us suppose that $\mathcal{H}$ is of dimension at least 3, and let $X \in \mathcal{C}(\mathcal{H})$, of dimension at least 2. If f is an ortholattice automorphism of $\mathcal{C}(\mathcal{H})$ whose restriction to $[0, X]$ is the identical map, there exists a unique unitary operator σ on $\mathcal{H}$ inducing f such that the restriction of σ to X is the identical map.

The following result can easily be expressed in terms of ortholattice theory, and then it provides a new ortholattice-theoretic version of Solèr's Theorem.

Theorem 1. The following two statements are equivalent:

(1) There exist $X, Y \in \mathcal{C}(\mathcal{H})$, where Y is of dimension at least 2, and an ortholattice automorphism f of $\mathcal{C}(\mathcal{H})$, such that $f(X)$ is strictly contained in X and the restriction of f to $[0, Y]$ is the identical map.

(2) $\mathcal{H}$ is an infinite-dimensional classical Hilbert space.

Moreover, if statement (2) holds, then, in order to prove that statement (1) holds, subspaces X and Y can be chosen in a quite arbitrary way: we need only have $\dim(Y) \geq 2$, $X \perp Y$, and X, $X^{\perp} \cap Y^{\perp}$ both infinite-dimensional.

Sketch of proof. (1) $\Rightarrow$ (2) By Lemma 1, there exists a unitary map σ: $\mathcal{H} \mapsto \mathcal{H}$ inducing f. If x_0 is any nonzero element of $f(X)^{\perp} \cap X$, then the sequence $(\sigma^n(x_0))_{n \in \mathbf{N}}$ is a γ-orthogonal system, where $\gamma = \langle x_0, x_0 \rangle$; thus we need only apply Solèr's Theorem.

(2) $\Rightarrow$ (1). Let $X, Y \varepsilon \mathcal{C}(\mathcal{H})$ satisfying the requirements of the last part of Theorem 1, and let $X_1, Y_1 \in \mathcal{C}(\mathcal{H})$ be separable infinite-dimensional subspaces resp. of X and $X^{\perp} \cap Y^{\perp}$.

There exists a Hilbert basis $(e_n)_{n \in \mathbf{Z}}$ (indexed by the set $\mathbf{Z}$ of all integers) of $X_1 + Y_1$ such that $(e_n)_{n \in \mathbf{N}}$ is a Hilbert basis of X_1. There is an unique unitary map σ: $\mathcal{H} \mapsto \mathcal{H}$ satisfying the following two conditions:

- $\sigma(e_n) = e_{n+1}$ for any $n \in \mathbf{Z}$.
- The restriction of σ to $(X_1 + Y_1)^{\perp}$ is the identity.

Then the ortholattice automorphism f of $\mathcal{C}(\mathcal{H})$ induced by σ satisfies all the requirements of statement (1).

4. COMPLETE CHARACTERIZATIONS OF THE UNDERLYING DIVISION RING

Let us denote by $C(K)$ the center of K. Let us define

$$U(K) := \{\gamma \in K | \gamma\gamma^* = 1\}$$

$$C_1(K) := C(K) \cap U(K)$$

and, for any nonzero element x_0 of $\mathcal{H}$,

$$V(x_0) := \{\gamma \in K | \langle \gamma x_0, \gamma x_0 \rangle = \langle x_0, x_0 \rangle\}$$

$$:= \{\gamma \in K | \gamma \langle x_0, x_0 \rangle \gamma^* = \langle x_0, x_0 \rangle\}$$

The sets $C_1(K)$ and $V(x_0)$ are both multiplicative subgroups of $K \setminus \{0\}$, and in any classical case, they are isomorphic to one of the classical groups $O(1)$, $SO(2)$, $SU(2)$.

More precisely, for $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$:

- $C_1(K)$ is isomorphic resp. to $O(1)$, $SO(2)$, $O(1)$.
- $V(x_0)$ is isomorphic resp. to $O(1)$, $SO(2)$, $SU(2)$.

Lemma 2. (a) If $\mathcal{H}$ is of dimension at least 2, a unitary map $\sigma: \mathcal{H} \mapsto \mathcal{H}$ induces the identical map on $\mathcal{C}(\mathcal{H})$ if and only if it is of the form $\gamma id_{\mathcal{H}}$, where $\gamma \in C_1(K)$. It follows that two unitary maps $\sigma_1, \sigma_2: \mathcal{H} \mapsto \mathcal{H}'$ induce the same automorphism f: $\mathcal{C}(\mathcal{H}) \mapsto \mathcal{C}(\mathcal{H}')$ if and only if there exists $\gamma \in C_1(K)$ such that $\sigma_2 = \gamma\sigma_1$.

(b) If $\mathcal{H}$ is one-dimensional, let x_0 be a nonzero fixed vector in $\mathcal{H}$. Then a map σ: $\mathcal{H} \mapsto \mathcal{H}$ is unitary if and only if there exists γ in $V(x_0)$ such that, for any $\lambda \in K$, $\sigma(\lambda x_0) = \lambda\gamma x_0$.

The statement (1) in the following theorem is connected with some results of Mayet and Pulmannová (1994) about nearly orthosymmetric ortholattices.

Theorem 2. Let $X \in \mathcal{C}(\mathcal{H})$, and let G be the group of all ortholattice automorphisms f of $\mathcal{C}(\mathcal{H})$ such that the restrictions of f to $[0, X]$ and to $[0, X^{\perp}]$ are both identical maps. Then:

(1) If X and $X^{\perp}$ are of dimension at least 2 (which implies that $\mathcal{H}$ is of dimension at least 4), G is isomorphic to $C_1(K)$.

(2) If X is one-dimensional and $X^{\perp}$ of dimension at least 2 (hence $\mathcal{H}$ is of dimension at least 3), the group G is isomorphic to $V(x_0)$, where x_0 is any nonzero vector of X.

Sketch of Proof. (1) For any $\gamma \in C_1(K)$, $x \in X$, $y \in X^{\perp}$, we define $\sigma_{\gamma}(x + y)$: $= \gamma x + y$.

(2) For any $\gamma \in V(x_0)$, $\lambda \in K$, $y \in X^{\perp}$, we define $\sigma_{\gamma}(\lambda x_0 + y) := \lambda\gamma^* x_0 + y$.

In both cases, by Lemmas 1 and 2, σ_{γ} is a unitary operator on $\mathcal{H}$, and if we denote by $\phi(\gamma)$ the automorphism of $\mathcal{C}(\mathcal{H})$ induced by σ_{γ}, then ϕ is a group isomorphism fro $C_1(K)$ [resp. $V(x_0)$] onto G.

Putting together Theorems 1 and 2, we obtain ortholattice-theoretic characterizations of the underlying division ring of a Hilbert lattice by means of automorphisms for each of the classical cases $K = \mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$.

Statement (1) in Theorem 2 allows us only to give such a characterization for $K = \mathbf{C}$ [since $C_1(\mathbf{R}) = C_1(\mathbf{H}) = \{-1, 1\}$]. If $\mathcal{H}$ is a classical Hilbert space, then, applying statement (1), we obtain that $\mathcal{H}$ is a complex Hilbert space if and only if there exists $g \in G$ such that $g^2 \neq id_{\mathcal{H}}$, and the next result follows:

Theorem 3. If $\mathcal{L}$ is any Hilbert lattice, the following two statements are equivalent:

(1) There exist pairwise orthogonal elements X, Y, Z of $\mathcal{L}$, where X, Y are of height at least 3, and an ortolattice automorphism f of $\mathcal{L}$ such that:

- The restrictions of f to X and Y are both identical maps.
- The restriction of f to $[0, X \vee Y]$ is not involutive.
- $f(Z)$ is strictly contained in Z.

(2) There exists an infinite-dimensional *complex* Hilbert space $\mathcal{H}$ such that $\mathcal{L}$ is isomorphic to $\mathcal{C}(\mathcal{H})$.

In a similar way, Theorem 1 together with statement (2) in Theorem 2 enable us to give complete characterizations of the underlying division ring for each of the classical cases, since then the corresponding groups G are pairwise nonisomorphic.

For instance, applying statement (2), we can express that the equation $\sigma^4 = Id_{\mathcal{H}}$ has respectively, two, four, or infinitely many solutions in G according to whether K is **R**, **C**, or **H**. Another way is to express that $K =$ **R** (resp. $K =$ **H**) is the only classical case where G is 2-element (resp. noncommutative).

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